Lebesgue Theory

Goal: To develop measure theory and build the Lebesgue integral as a more general integral than, say, the Riemann integral.

A motivating example:

Consider the nowhere continuous function

$$f : [0,1] \to \mathbb{R}$$

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1] \\ 0, & \text{otherwise} \end{cases}$$

f is not Riemann integrable

(because partitioning $[0,1]$ into intervals, each interval will have at least one $x \in \mathbb{Q}$ & one $x \notin \mathbb{Q}$ so the upper and lower Darboux sums will be 1 & 0 respectively.)

On the other hand, "morally" this integral, i.e., the area under the graph of f should be zero.

Reasoning: Since $[0,1] \cap \mathbb{Q}$ is countable, we can enumerate its elements, say $q_1, q_2, \ldots$,

Fix $\varepsilon > 0$,

Now, put an interval I_i of size $\frac{\varepsilon}{2^i}$ around each q_i.

& define $f_\varepsilon(x) = \begin{cases} 1, & x \in I_i \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$

So $0 \leq f \leq f_\varepsilon$

If we want the "area" under the graph of f_ε

$$\text{"Area"}_\varepsilon = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} \cdot 1 = \varepsilon$$

Taking limits $\text{"Area"}_\varepsilon \to 0$ so f should be integrable and the integral should be zero.

Remark: In our construction $\bigcup I_i \neq [0,1]$ and $[0,1] \setminus (\bigcup_i I_i)$ contains no intervals (since it contains no rationals)

To make sense of all this, we'll need a theory of measure

Set Functions:

Notation: If A & B are two sets

$$A \setminus B = \{x : x \in A, x \notin B\}$$

$\emptyset$ = empty set

Ring: A family $\mathcal{R}$ of sets is called a ring if

$$A \in \mathcal{R}, B \in \mathcal{R} \Rightarrow A \cup B \in \mathcal{R}$$
$$A \setminus B \in \mathcal{R}$$

(so a ring is closed under finite unions & set differences)

Remark: $A \cap B = A \setminus (A \setminus B)$ so a ring is also closed under finite intersections

σ-ring: is a ring where

$$A_n \in \mathcal{R}, n = 1, 2, \ldots \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$$

(closed under countable unions)

Remark: $A_1 \setminus A_n \in \mathcal{R} \ \forall n = 1, 2, \ldots$

$$\Rightarrow \bigcup_{n=1}^{\infty} (A_1 \setminus A_n) \in \mathcal{R}$$

$$\Rightarrow A_1 \setminus \bigcup_{n=1}^{\infty} (A_1 \setminus A_n) \in \mathcal{R}$$

$$\Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{R}$$

Definition: • A set function defined on $\mathcal{R}$ assigns to every $A \in \mathcal{R}$ a number (in the extended reals)

• A set function f is additive if

$$A \cap B = \emptyset \Rightarrow f(A \cup B) = f(A) + f(B)$$

need to assume that either $+\infty$ or $-\infty$ $\notin$ range f.

• It is countably additive if

$$A_i \cap A_j = \emptyset \ \forall i \neq j \Rightarrow f\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} f(A_n)$$

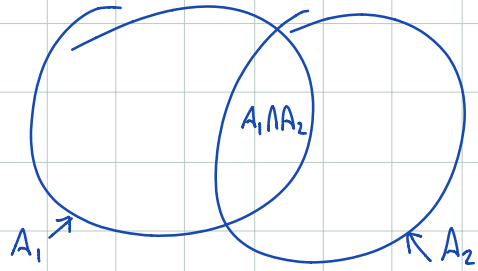
independent of order $\Rightarrow$ converges absolutely when it does.

Some Properties of additive set functions

- $f(\phi) = 0$ bec $\phi \cap \phi = \phi \Rightarrow f(\phi \cup \phi) = f(\phi) + f(\phi)$

- $f(A_1 \cup \dots \cup A_n) = f(A_1) + f(A_2) + \dots + f(A_n)$
  when $A_i \cap A_j = \phi \quad \forall i \neq j$

- $f(A_1 \cup A_2) + f(A_1 \cap A_2) = f(A_1) + f(A_2)$



- If $f(A) \geq 0 \quad \forall A$ & $A_1 \subset A_2$ then $f(A_1) \leq f(A_2)$

  (why?)

- If $B \subset A$ & $|f(B)| < \infty$ then $f(A \setminus B) = f(A) - f(B)$

  (why?)

Theorem: Let μ be a countably additive function on a ring R. Suppose $A_n \in R$ for $n = 1, 2, \ldots$ & $A_1 \subset A_2 \subset \cdots$ Suppose that

$$A := \bigcup_{n=1}^{\infty} A_n \in R.$$

Then $\lim_{n \to \infty} \mu(A_n) = \mu(A)$.

Proof: The idea is that we want to find disjoint sets B_n with $\bigcup_n B_n = A$

so that we can use countable additivity.

Specifically, take $B_1 = A_1$

$$B_2 = A_2 \setminus A_1$$

$$\vdots$$

$$B_n = A_n \setminus A_{n-1}$$

then $\bigcup_{n=1}^{m} B_n = A_1 \cup \left(\bigcup_{n=2}^{m} (A_n \setminus A_{n-1}) \right) = A_m$

& $B_i \cap B_j = \emptyset \quad \forall i \neq j$

& $\bigcup_{n=1}^{\infty} B_n = A$

So $F(A_n) = \sum_{i=1}^{n} F(B_i)$

& $\lim_{n\to\infty} F(A_n) = \sum_{i=1}^{\infty} F(B_i) = F(\dot{\bigcup}_{i=1}^{\infty} B_i) = F(A)$

↑ by countable additivity

∎

(Slow) Construction of the Lebesgue Measure

Interval in $\mathbb{R}^p$: $[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_p \times b_p]$

(note that any (or all) of the intervals $[a_i, b_i]$ can be replaced by (a_i, b_i) or $(a_i, b_i]$ or $[a_i, b_i)$.)

Elementary set: A finite union of intervals

- For an interval I, define

$$m(I) = \prod_{i=1}^{p} (b_i - a_i)$$

- For $A = \bigcup_{i=1}^{n} I_i$, where $I_i \cap I_j = \phi \ \forall i \neq j$

  set $m(A) = \sum_{i=1}^{n} m(I_i)$

- Let $\mathcal{E}$ be the family of elementary subsets of $\mathbb{R}^p$

Properties of $\mathcal{E}$

- $\mathcal{E}$ is a ring (closed under finite union and set differences)
- $\mathcal{E}$ is not a σ-ring (not closed under countable unions/intersections)
- $A \in \mathcal{E} \Rightarrow A = \bigcup_{i=1}^{n} I_i$ for some finite n & disjoint intervals A_i

  (If the intervals are not disjoint, just decompose them)

- $A \in \mathcal{E}$: If $A = \bigcup_{i=1}^{n} I_i$ & $A = \bigcup_{i=1}^{m} J_i$

where I_i's are disjoint intervals
J_i's are disjoint intervals

then $m(A) = \sum_{i=1}^{n} m(I_i) = \sum_{i=1}^{m} m(J_i)$

- m is additive on $\mathcal{E}$

Definition: (Regularity)

A nonnegative, additive set function F defined on $\mathcal{E}$ is regular if

$\forall A \in \mathcal{E}$, & $\epsilon > 0$, $\exists F_\epsilon, G_\epsilon \in \mathcal{E}$ where F is closed & G is open &

$F_\epsilon \subset A \subset G_\epsilon$ (closed → F_ϵ, G_ϵ ← open; ← regular F)

with $F(G_\epsilon) - \epsilon \leq F(A) \leq F(F_\epsilon) + \epsilon$

Example: • on Intervals $m(I)$ is regular (can always find closed and open subintervals satisfying the def'n.)

• on elementary sets $m(I)$ is regular because elementary sets can be written as disjoint unions of intervals.

Next Step: Show that every regular set function on $\mathcal{E}$ can be extended to a countably additive set function on a σ-ring containing $\mathcal{E}$.

Definition: Let μ be an additive, non-negative, finite, regular set-function on $\mathcal{E}$.

Define the outer measure of $E \subset \mathbb{R}^p$

$$\mu^*(E) = \inf \sum_{n=1}^{\infty} \mu(A_n)$$

where the infimum is taken over all countable coverings of E by elementary (open) sets, i.e.,

$$E \subset \bigcup_{n=1}^{\infty} A_n, \quad A_n \in \mathcal{E}$$

Properties: • $\mu^*(E) \geq 0$ (since $\mu \geq 0$)

$$E_1 \subset E_2 \Rightarrow \mu^*(E_1) \subset \mu^*(E_2)$$

(since any cover for E_2 is a cover for E_1)

Theorem: Let μ be finite, non-negative, additive regular

(a) $A \in \mathcal{E} \Rightarrow \mu^*(A) = \mu(A)$

(b) $E = \bigcup_{i=1}^{\infty} E_i \Rightarrow \mu^*(E) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$

subadditivity

Proof: (a) Let $A \in \mathcal{E}$

- By regularity, $\exists G \in \mathcal{E}$, open s.t. $A \subset G$

& $\mu(G) - \varepsilon \leq \mu(A) \Rightarrow \mu(G) \leq \mu(A) + \varepsilon$

but $\mu^*(A) \leq \mu(G)$ and ε is arbitrary $\Rightarrow \mu^*(A) \leq \mu(A)$ — (a)

- But $\mu^*(A) = \inf \sum_{n=1}^{\infty} \mu(A_n)$ (by def'n)

where $A_n \in \mathcal{E}$, open, & $A \subset \bigcup_{n=1}^{\infty} A_n$

$\Rightarrow \exists$ a sequence $\{A_n\}$ of open, elementary sets s.t.

$$\sum_{n=1}^{\infty} \mu(A_n) \leq \mu^*(A) + \varepsilon \quad \text{— (1)}$$

By regularity of μ, $\exists F \subset A$, closed

with $\mu(A) \leq \mu(F) + \varepsilon$ — (2)

Moreover $F \subset A \in \mathcal{E} \Rightarrow F$ is compact

$\Rightarrow F \subset A_1 \cup A_2 \cup \dots \cup A_N$ for a finite N

$\Rightarrow$

$$\mu(A) \underset{\text{by (2)}}{\leq} \mu(F) + \varepsilon \overset{F \subset \bigcup_{i=1}^{N} A_i}{\leq} \mu\left(\bigcup_{i=1}^{N} A_i\right) + \varepsilon$$

$$\leq \sum_{i=1}^{N} \mu(A_i) + \varepsilon$$

$$\underset{\text{by (1)}}{\leq} \mu^*(A) + \varepsilon + \varepsilon$$

Hence $\mu(A) \leq \mu^*(A) + 2\varepsilon$

but $\mu^*(A) \leq \mu(A)$ (by (0))

As this is true $\forall \varepsilon$ $\mu(A) = \mu^*(A)$

Proof: (b)

Suppose $E = \bigcup_{n=1}^{\infty} E_n$ and assume $\mu^*(E_n) < +\infty$ $\forall n$ (otherwise the conclusion is trivially true)

Fix $\varepsilon > 0$

Each E_n has a covering $\{A_{nk}\}$ by open elementary sets such that

$$\sum_{k=1}^{\infty} \mu(A_{nk}) \leq \mu^*(E_n) + 2^{-n}\varepsilon$$

↑ by def'n of μ^*

Since $\bigcup_{k,n} A_{nk}$ covers E, & $\mu^*(E)$ is the inf.

$$\Rightarrow \quad \mu^*(E) \leq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \mu(A_{nk})$$

$$\leq \sum_{n=1}^{\infty} \left(\mu^*(E_n) + 2^{-n}\varepsilon\right)$$

$$= \sum_{n=1}^{\infty} \mu^*(E_n) + \varepsilon$$

$\Rightarrow$ as ε was arbitrary $\mu^*(E) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$

∎

Restatement of the theorem

as the outer measure is defined for any $E \subset \mathbb{R}^p$, & and as $\mu^*(A) = \mu(A)$ when A is elementary, part (a) states that μ^* is an extension of μ from $\mathcal{E}$ to all subsets of $\mathbb{R}^p$.

part (b) states that μ^* is sub-additive.

Definitions: • $S(A,B) = (A \setminus B) \cup (B \setminus A)$ — symmetric set diff

• $d(A,B) = \mu^*(S(A,B))$ — "distance"

we write $A_n \to A$ if $\lim_{n\to\infty} d(A, A_n) = 0$

- If $\exists \{A_n\}$, $A_n \in \mathcal{E}$, $A_n \to A$

we say that A is finitely μ-measurable

and we write $A \in \mathcal{M}_F(\mu)$

- If $A = \bigcup_{n=1}^{\infty} A_n$ where $A_n \in \mathcal{M}_F(\mu)$

we say that A is μ-measurable & write

$$A \in \mathcal{M}(\mu)$$

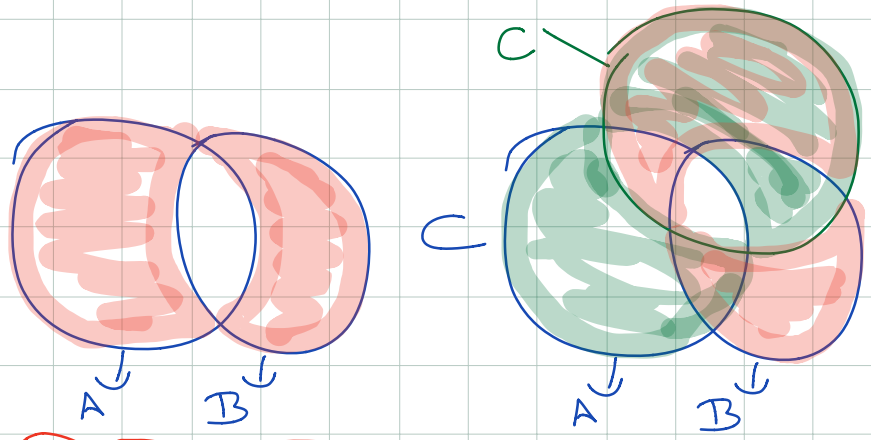
Theorem: $\mathcal{M}(\mu)$ is a σ-ring & μ^* is countably additive on $\mathcal{M}(\mu)$

Detour: to prove this theorem, we will need some properties of $S(A,B)$ and $d(A,B) = \mu^*(S(A,B))$

- $S(A,B) = S(B,A)$ & $S(A,A) = \phi$

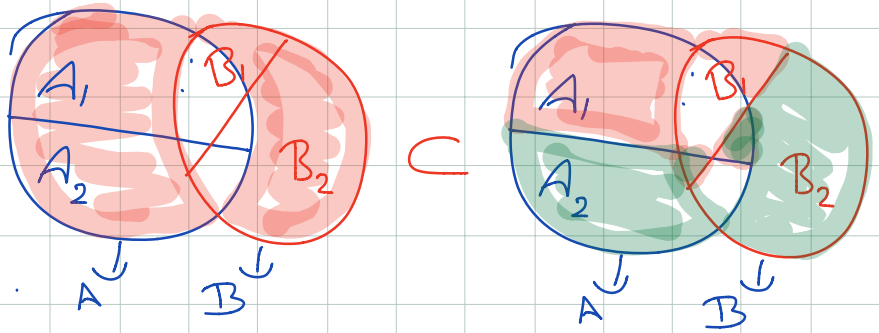
$\Rightarrow$ $d(A,B) = d(B,A)$ & $d(A,A) = 0$ — (P1)

- $S(A,B) \subset S(A,C) \cup S(C,B)$



$\Rightarrow$ $d(A,B) \leq d(A,C) + d(C,B)$ — (P2) (subadditivity of $|\cdot|$)

- $S(A_1 \cup A_2, B_1 \cup B_2) \subset S(A_1,B_1) \cup S(A_2,B_2)$



$\Rightarrow$ $d(A_1 \cup A_2, B_1 \cup B_2) \leq d(A_1,B_1) + d(A_2,B_2)$

(P3a)

Similarly.

- $S(A_1 \cap A_2, B_1 \cap B_2) \subset S(A_1, B_1) \cup S(A_2, B_2)$

$$\Rightarrow d(A_1 \cap A_2, B_1 \cap B_2) \leq d(A_1, B_1) + d(A_2, B_2)$$

(P3b)

- $S(A_1 \setminus A_2, B_1 \setminus B_2) \subset S(A_1, B_1) \cup S(A_2, B_2)$

$$\Rightarrow d(A_1 \cap A_2, B_1 \cap B_2) \leq d(A_1, B_1) + d(A_2, B_2)$$

(P3c)

Remark: P1a & P2a make d almost a metric except that $d(A,B) = 0$ doesn't imply $A = B$

Proof: Take $\mu = m$, $A = \{x_n, n = 1, 2, \ldots\}$, $B = \emptyset$

$$d(A,B) = m^*(S(A, \emptyset)) = m^*(A)$$

$$= \inf_{A_n} \sum_{n=1}^{\infty} m(A_n) \text{ where } A_n \in \mathcal{E}, \text{ open}, \bigcup_{n=1}^{\infty} A_n \supset A$$

Take A_n s.t. $x_n \in A_n$, $m(A_n) < 2^{-n}\varepsilon$

$\Rightarrow m^*(A) < \sum_n 2^{-n}\varepsilon = \varepsilon \leftarrow$ arbitrary

$\Rightarrow d(A,B) = 0$ while $A \neq B$.

To make d a metric, Let us consider equivalence classes defined by $d(A,B)=0$
(so $B \in [A] \Leftrightarrow d(A,B)=0$)

Now the set of equivalence classes is a metric space with d as the metric.

Lemma: Suppose $\mu^*(B) < \infty$ or $\mu^*(A) < \infty$
then $|\mu^*(A) - \mu^*(B)| \leq d(A,B)$

Proof: Suppose $0 \leq \mu^*(B) \leq \mu^*(A)$
& $\mu^*(B) < \infty$

$d(A,\phi) \leq d(A,B) + d(B,\phi)$

$\mu^*(A) \leq d(A,B) + \mu^*(B)$

$\Rightarrow \mu^*(A) - \mu^*(B) \leq d(A,B)$

Back to our theorem :

Theorem

$\mathcal{M}(\mu)$ is a σ-ring & μ^* is countably additive on $\mathcal{M}(\mu)$

Proof :

Technique : (I) Show that $\mathcal{M}_F(\mu)$ is a ring & μ^* is additive on $\mathcal{M}_F(\mu)$

(II) Show that $A \in \mathcal{M}(\mu)$

$$\Rightarrow \mu^*(A) = \sum_{n=1}^{\infty} \mu^*(A_n)$$

(A_n disjoint, $\in \mathcal{M}_F(\mu)$)

& that $A \in \mathcal{M}(\mu)$, $\mu^*(A) < \infty$ $\Rightarrow A \in \mathcal{M}_F(\mu)$

to deduce μ^* is countably additive on $\mathcal{M}(\mu)$

(III) Show that $\mathcal{M}(\mu)$ is a ring.

Now, for the proof:

Step (I): Let $A, B \in \mathcal{M}_F(\mu)$

$\Rightarrow \exists\ A_n$ & $B_n \in \mathcal{E}$, $A_n \to A$, $B_n \to B$

$\Rightarrow d(A_n \cup B_n, A \cup B) \leq d(A_n, A) + d(B_n, B)$

$\longrightarrow 0$

So $A_n \cup B_n \to A \cup B$ — (1)

Similarly $A_n \cap B_n \to A \cap B$

$A_n \setminus B_n \to A \setminus B$ — (2)

(by P.3a, P3b, P3c)

Moreover by the lemma (3) $\mu^*(A_n) \to \mu^*(A)$

(bec $|\mu^*(A_n) - \mu^*(A)| \leq d(A_n, A) \to 0$)

$\mu^*(A) < \infty$, bec $A \in \mathcal{E}$

So $\mu^*(A) < \infty$. Moreover, by (1) & (2)

$\mathcal{M}_F(\mu)$ is a ring

($A_n \cup B_n \in \mathcal{E} \to A \cup B$, $A_n \setminus B_n \in \mathcal{E} \to A \setminus B$

So $A \cup B \in \mathcal{M}_F(\mu)$ & $A \setminus B \in \mathcal{M}_F(\mu)$)

Now, to show μ^* is additive on $\mathcal{M}_F(\mu)$:

$$\mu(A_n) + \mu(B_n) = \mu(A_n \cup B_n) + \mu(A_n \cap B_n)$$

Taking limits $n \to \infty$ and using (3)

$$\mu^*(A) + \mu^*(B) = \mu^*(A \cup B) + \mu^*(A \cap B)$$

Now, if $A \cap B = \phi$, $\mu^*(A) + \mu^*(B) = \mu^*(A \cup B)$

$\Rightarrow$

μ^* is additive on $\mathcal{M}_F(\mu)$.

Step (II) (Show that μ^* is countably additive on $\mathcal{M}(\mu)$)

Let $A \in \mathcal{M}(\mu) \Rightarrow A = \bigcup_n A_n'$, $A_n' \in \mathcal{M}_F(\mu)$

Let $A_1 = A_1'$

$A_2 = (A_1' \cup A_2') \setminus (A_1')$ (so $A_1 \cap A_2 = \phi$ & $A_1 \cup A_2 = A_1' \cup A_2'$)

$A_3 = (A_1' \cup A_2' \cup A_3') \setminus (A_1' \cup A_2')$

(so $A_i \cap A_j = \phi$ $i \neq j$ & $\bigcup_{i=1}^{3} A_i = \bigcup_{i=1}^{3} A_i'$)

& set

$$A_n = \left(\bigcup_{i=1}^{n} A_i\right) \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)$$

So $A \in \mathcal{M}(\mu) \Rightarrow A = \bigcup_{n=1}^{\infty} A_n$, A_n's are disjoint $A_n \in \mathcal{M}_F(\mu)$

By subadditivity of μ^*:

$$\mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n) \quad (4)$$

By additivity of μ^* on $\mathcal{M}_F(A)$ & the fact that $\bigcup_{i=1}^{n} A_i \subset A$:

$$\mu^*(A) \geq \sum_{i=1}^{n} \mu^*(A_i) \quad (5)$$

(4) & (5) $\Rightarrow$

$$\mu^*(A) = \sum_{n=1}^{\infty} \mu^*(A_n) \quad (6)$$

where A_n's disjoint, A_n's $\in \mathcal{M}_F(A)$

This doesn't complete the proof of additivity on $\mathcal{M}(\mu)$ as the A_n's were in $\mathcal{M}_F(\mu)$, not $\mathcal{M}(\mu)$.

Now, suppose $\mu^*(A) < \infty$

and set $B_n = \bigcup_{i=1}^{n} A_i \subset A$

By ⑥: $d(A, B_n) = \mu^*\left(\bigcup_{i=n+1}^{\infty} A_i\right)$

$$= \sum_{i=n+1}^{\infty} \mu^*(A_i) \xrightarrow{n\to\infty} 0$$

So $B_n \to A$ but $B_n \in \mathcal{M}_F(\mu)$

($\mathcal{M}_F(\mu)$ is a ring)

So $A \in \mathcal{M}_F(\mu)$ when $A \in \mathcal{M}(\mu)$ & $\mu(A) < \infty$

To show that μ^* is countably additive on $\mathcal{M}(\mu)$, suppose that

$A = \bigcup_{n=1}^{\infty} A_n$ where A_n's are disjoint & $A_n \in \mathcal{M}(\mu)$.

- If $\mu^*(A_n) < \infty \ \forall n$, i.e. every $A_n \in \mathcal{M}_F(\mu)$ and by (6): $\mu^*(A) = \sum_{n=1}^{\infty} \mu^*(A_n)$

- Otherwise if $\mu^*(A_n)$ is infinite, & $A_n \subset A$ $\mu^*(A)$ is infinity

So μ^* is countably additive on $\mathcal{M}(\mu)$

Step III (Show that $\mathcal{M}(\mu)$ is a σ-ring)

Recall that $A \in \mathcal{M}(\mu)$ if it is a countable union of E_n's, $E_n \in \mathcal{M}_F(\mu)$.

So if $A_n \in \mathcal{M}(\mu) \ \forall n$

then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}(\mu)$ (countable union of countable unions is countable)

So we have closure under countable unions

Remains to show closure under set differences

Let $A \in \mathcal{M}(\mu)$ with $A = \bigcup_{n=1}^{\infty} A_n$

$B \in \mathcal{M}(\mu)$ with $B = \bigcup_{n=1}^{\infty} B_n$

where A_n's, B_n's $\in \mathcal{M}_F(\mu)$

But $A_n \cap B = \bigcup_{i=1}^{\infty} (\underbrace{A_n \cap B_i}_{\in \mathcal{M}_F(\mu)}) \Rightarrow A_n \cap B \in \mathcal{M}(\mu)$

Moreover $\mu^*(A_n \cap B) \leq \mu^*(A_n) < \infty$

$\Rightarrow A_n \cap B \in \mathcal{M}_F(\mu)$

So $A_n \setminus B \in \mathcal{M}_F(\mu)$

But $A \setminus B = \bigcup_{n=1}^{\infty} \underbrace{(A_n \setminus B)}_{\in \mathcal{M}_F(\mu)}$

$\Rightarrow A \setminus B \in \mathcal{M}(\mu)$

So we have a σ-ring

Remark: We can now simply write $\mu(A)$ instead of $\mu^*(A)$ whenever $A \in \mathcal{M}(\mu)$ as we have extended the additive function μ defined on $\mathcal{E}$ to a countably additive function on $\mathcal{M}(\mu)$

We call this extended function a measure

When $\mu = m$, we call it the Lebesgue measure on $\mathbb{R}^p$

Borel Sets

- E is a Borel set if it can be obtained by a countable union and/or (countable) intersections and/or complements of open sets.

- The collection of all Borel sets (in $\mathbb{R}^p$) is a σ-ring (why?). Call it $\mathcal{B}$.

- It is the smallest σ-ring containing all open sets.

Proof:

Suppose $\mathcal{B}'$ is a σ-ring that is smaller than $\mathcal{B}$ and contains all open sets in $\mathbb{R}^p$.

then $\exists A \in \mathcal{B}$, $A \notin \mathcal{B}'$

but $A \in \mathcal{B} \Rightarrow A$ is a countable union \ intersection \ comp. of open sets $\Rightarrow A \in \mathcal{B}'$ because $\mathcal{B}'$ is a σ-ring.

Lemma: If $A \in \mathcal{M}(\mu)$, $\exists$ Boul sets F & G, $F \subset A \subset G$ with

$$\mu(G \setminus A) = \mu(A \setminus F) = 0$$

Proof: If $A \in \mathcal{M}(\mu)$ & $\varepsilon > 0$, $\exists F, G$

F is closed & G is open st. $F \subset A \subset G$

and $\mu(G \setminus A) < \varepsilon$

bec $\mu^*(A) = \inf \mu\left(\bigcup_n A_n\right)$, A_n open, $A_n \in \mathcal{E}$, $\bigcup A_n \supset A$

$\| \mu(A)$

So $\exists$ open G : $\mu(G) < \mu(A) + \varepsilon$

$\Rightarrow \mu(G \setminus A) + \mu(A) < \mu(A) + \varepsilon$

and $\mu(A \setminus F) < \varepsilon$

(follows by taking complements).

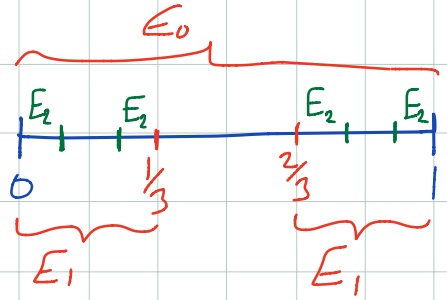
now, take $\varepsilon = \frac{1}{n}$ & $n \to \infty$ $\Rightarrow \mu(G \setminus A) = \mu(A \setminus F) = 0$ ∎

By the lemma every $A \in \mathcal{M}(\mu)$ is the union of a Borel set & a set of measure zero ($A = A \setminus F \cup F$, with $\mu(A \setminus F) = 0$ and F a Borel set).

Remarks:

- When $\mu = m$ is the Lebesgue measure, if A is countable $m(A) = 0$ (we saw this before)
- $\exists$ uncountable sets, e.g., the Cantor set, of measure zero.

Proof:



$\Rightarrow E_0 \supset E_1 \supset E_2 \cdots$, E_n = union of 2^n intervals, of length 3^{-n}

$$m(E_n) = \left(\frac{2}{3}\right)^n \quad \text{for } n = 1, 2, 3, \ldots$$

$$P = \bigcap_{\forall n} E_n \Rightarrow P \subset E_n \Rightarrow m(P) = 0.$$